Nutcracker: Physics Problem 1

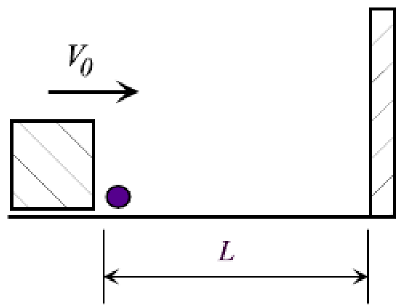
1 Problem- A Multitude of Collisions

A block with very large mass M slides on a frictionless surface towards a fixed wall. The block's speed is V_o. The block strikes a particle with very small mass m (and negligible size), which is initially at rest at a distance L from the wall. The particle bounces elastically off the block and slides to the wall, where it bounces elastically and then slides back towards the block. The particle continues to bounce elastically back and forth between the block and the wall.

(a) How close does the block come to the wall?

(b) How many times does the particle bounce off the block, by the time the block makes its closest approach to the wall?

In both parts (a) and (b), you may assume $M >> m$, and you need only obtain approximate answers, valid to leading order in m/M. There is no friction and ignore the possibilty of toppling of block.



2 Solution

(a) Let v and V (both towards right) be the velocities of m and M after a collision when the distance from the fixed wall is l. After some time they will collide again at a distance $l^{'}$ from the wall after which their velocities become $v^{'}$ and $V^{'}$ (again both towards right). Simple kinematics yields,

$$l^{'} = l\frac{v-V}{v+V}$$

Also the classic result of elastic collision is that the velocity of separation is same as the velocity of approach. Thus,

$$v + V = v^{'} - V'$$

This yields the invariant,

$$l(v - V) = l^{'}(v' - V')$$

Initially the value of this invariant is= LV_o

Since energy is conserved, velocity of the m when M stops*= $V_o\sqrt{\frac{M}{m}}$

Thus if L_c be the distance of closest approach then,

$$L_cV_o\sqrt{\frac{M}{m}} \approx LV_o$$

$$L_c \approx L\sqrt{\frac{m}{M}}$$

(b) It is easy to see that,

$$\begin{pmatrix} V' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{M-m}{M+m} & \frac{-2m}{M+m} \\ \frac{2M}{M+m} & \frac{M-m}{M+m} \end{pmatrix} \begin{pmatrix} V \\ v \end{pmatrix}$$

The eigen-values and eigen-vectors of the matrix are:

$$A_+ = \begin{pmatrix} \sqrt{\frac{m}{M}} \\ \frac{1}{i} \end{pmatrix}, \lambda_+ = \frac{M-m}{M+m} + i\frac{2\sqrt{Mm}}{M+m} = e^{i\phi}$$

$$A_- = \begin{pmatrix} \sqrt{\frac{m}{M}} \\ -\frac{1}{i} \end{pmatrix}, \lambda_- = \frac{M-m}{M+m} - i\frac{2\sqrt{Mm}}{M+m} = e^{-i\phi}$$

*It must be remembered that *the block stops* is an approximation in the limit $M >> m$. We shall see in the next part that it is really not necessary that M comes to rest- it might change the direction of velocity without even coming to rest. But in the limit $M >> m$, M indeed attains a very small speed tending to zero before its direction of velocity is reversed.

where, $\tan\phi = \dfrac{2\sqrt{Mm}}{M-m}$

The initial conditions are $(V, v) = (V_o, 0) = \dfrac{V_o}{2}\sqrt{\dfrac{M}{m}}(A_+ + A_-)$. Thus the vector (V_n, v_n) after nth collision will be

$$\begin{pmatrix} V_n \\ v_n \end{pmatrix} = \frac{V_o}{2}\sqrt{\frac{M}{m}}(\lambda_+^n A_+ + \lambda_-^n A_-)$$

which after some manipulation yields,

$$\begin{pmatrix} V_n \\ v_n \end{pmatrix} = \begin{pmatrix} V_o \cos n\phi \\ V_o\sqrt{\frac{M}{m}} \sin n\phi \end{pmatrix}$$

If N be the number of collisions made till the distance of closest approach then it is trivial to notice that setting $V_N = 0$ yields the value for $N^\dagger$ as,

$$N\phi \approx \frac{\pi}{2}$$

Thus the required number of collisions are,

$$N \approx \frac{\pi}{2\tan^{-1}\frac{2\sqrt{Mm}}{M-m}}$$

which in the given limit $M >> m$ becomes,

$$N \approx \frac{\pi}{4}\sqrt{\frac{M}{m}}$$

†Strictly speaking for $M > m$, N can be exactly determined to be

$$N = \left\lceil \frac{\pi}{2\phi} \right\rceil$$

where $\lceil x \rceil$ is the ceiling function for x